미래기반 가속기
전문인력양성 사업단

특론: 가속기 실험실습 I

(NUCE719P-01/PHYS715P-01, 정모세)

eLABs 시설을 이용한 빔운전 및 RF/빔진단 기초 2
(부제: RF 기초 3)

정모세

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Waveguides and Cavities

(Sec. 5.2 and Sec. 5.3 of UP-ALP)

Plane waves in free space

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- For free space (no boundary):

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

- Phase velocity:

$$v_{ph} = \frac{\omega}{|\mathbf{k}|} = \frac{\omega}{\sqrt{k_x^2 + k_y^2 + k_z^2}} = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

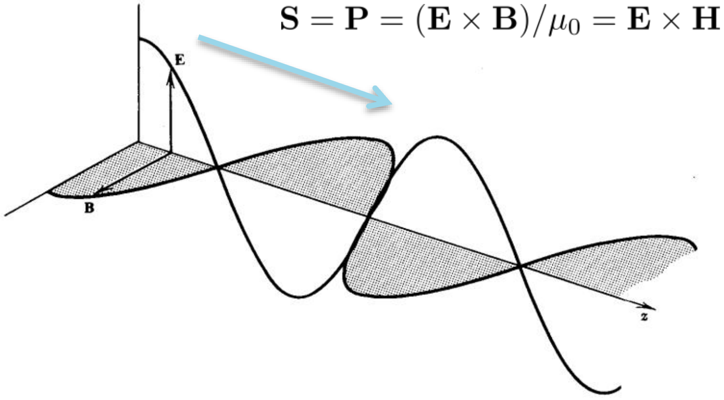
- Consequences of Maxwell equations:

$$\nabla \cdot \mathbf{E} = 0 \longrightarrow \mathbf{k}\cdot\mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0 \longrightarrow \mathbf{k}\cdot\mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \longrightarrow \mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \quad \nabla \times \mathbf{B} = -i\omega\mu_0\epsilon_0\mathbf{E} \longrightarrow \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}$$

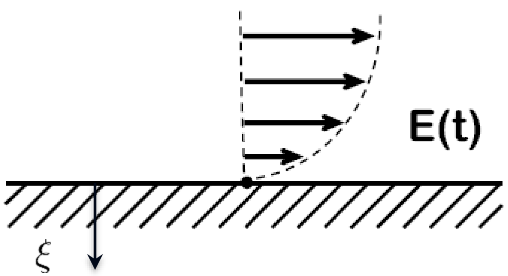
$$\boxed{\mathbf{E}\perp\mathbf{k}, \quad \mathbf{B}\perp\mathbf{k}, \quad \mathbf{E}\perp\mathbf{B}, \quad E = cB}$$

$$\mathbf{S} = \mathbf{P} = (\mathbf{E}\times\mathbf{B})/\mu_0 = \mathbf{E}\times\mathbf{H}$$

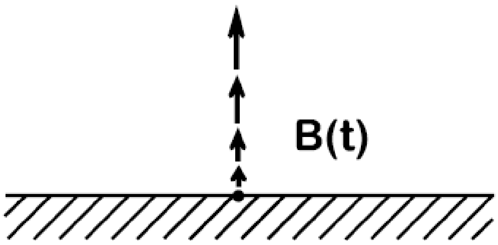


Boundary conditions at conducting surfaces

- On the surface of a perfect conductor, the tangential component of an electric field and the normal component of a magnetic field will vanish.



$$E_{||} = 0$$



$$B_{\perp} = 0$$

- A non-ideal surface has a finite conductivity (σ):

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2} = \frac{1}{\sqrt{\pi f \mu_0 \mu_r \sigma}},$$

Skin depth

Time-averaged power absorbed per unit area

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2} \times \underbrace{\left(\frac{1}{\sigma\delta}\right)}_{=R_{surf} = \sqrt{\mu_c \omega / 2\sigma} = \text{Surface resistance}} |\mathbf{K}_{\text{eff}}|^2$$

$$\mathbf{J} = \sigma \mathbf{E}_c, \;\; \mathbf{K}_{\text{eff}} = \int_0^{\infty} \mathbf{J} d\xi = \hat{\mathbf{n}} \times \mathbf{H}_{||}$$

Electric field in the conductor

Effective surface current density

Group velocity

- Interference between two continuous waves slightly different frequencies and wavenumbers:

$$
\begin{aligned}
E &= E_1 + E_2 \\
&= E_0 \sin\left[(k+dk)x - (\omega + d\omega)t\right] + E_0 \sin\left[(k-dk)x - (\omega - d\omega)t\right] \\
&= 2E_0 \sin[kx - \omega t] \cos[dk\ x - d\omega\ t] \\
&= 2E_0 f_1(x,t) f_2(x,t)
\end{aligned}
$$

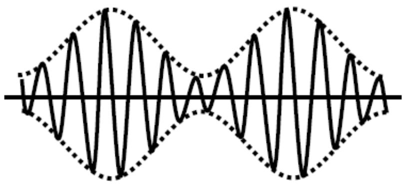


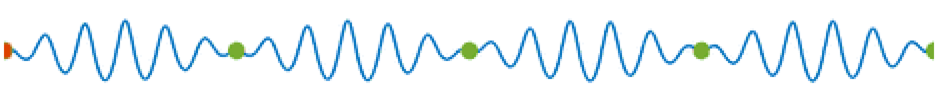
FIGURE 5.14
Two-wave interference.

- Phase velocity: by requesting the convective derivative of f_1 to be equal to zero

$$0 = \left(\frac{\partial}{\partial t} + v_p \frac{\partial}{\partial x}\right) f_1 \rightarrow v_p = -\frac{\partial f_1(x,t)/\partial t}{\partial f_1(x,t)/\partial x} = \frac{\omega}{k}$$

- Group velocity: by requesting the convective derivative of f_2 to be equal to zero

$$0 = \left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) f_2 \rightarrow v_g = -\frac{\partial f_2(x,t)/\partial t}{\partial f_2(x,t)/\partial x} = \frac{d\omega}{dk}$$

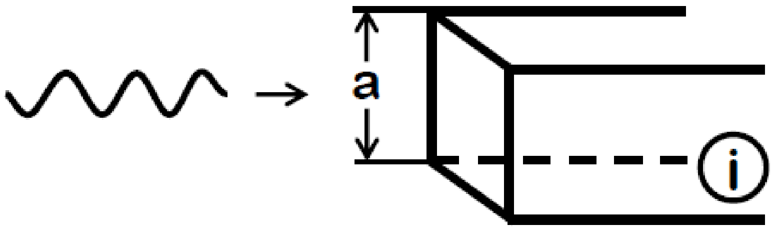


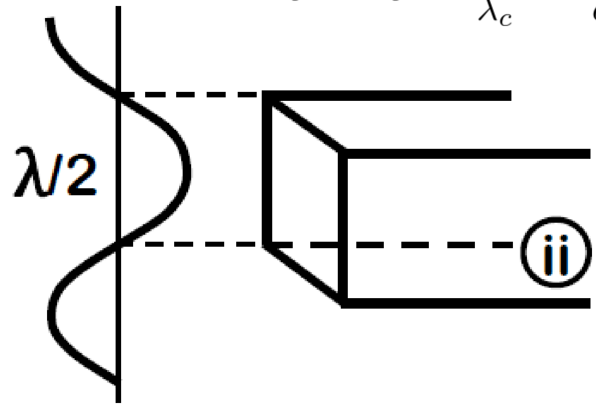
- The red square moves with the phase velocity
- The green circles propagate with the group velocity

Dispersion for a waveguide (Qualitative)

$$\lambda \ll a$$
$$\omega = ck$$

$$\lambda/2 \le a, \quad \text{or} \quad \lambda_c = 2a$$
$$\omega_c = ck_c = c\frac{2\pi}{\lambda_c} = c\frac{\pi}{a}$$



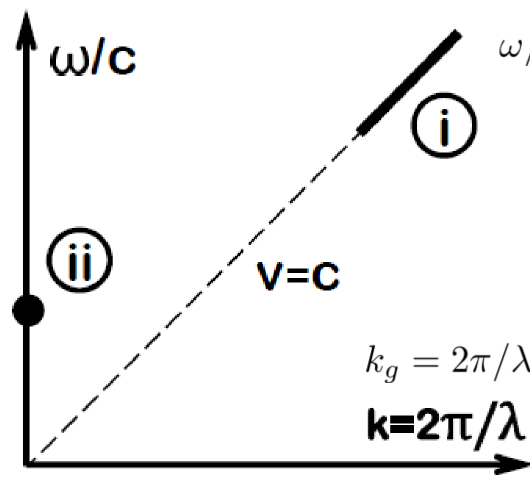


If the wavelength of an EM wave in free space is much shorter than the transverse size "*a*" of the waveguide then the waveguide does not matter.

When half of a wavelength in free space equals the waveguide transverse size, that is the longest wavelength for which the boundary conditions at a perfectly conducting surface of the waveguide can still be satisfied.

$$\omega/c = k$$

$$\omega_c/c = k_c = \frac{2\pi}{\lambda_c} = \frac{\pi}{a}$$



$$k_g = 2\pi/\lambda_g$$

TM Mode Solution in Circular Waveguide

- From conducting boundary, electromagnetic wave can be transformed into TM (Magnetic field is Transverse to z) mode.
- TM fields can be found from one vector component of the magnetic vector potential (note that $\nabla \cdot \mathbf{A} \neq 0$, i.e. we are using Lorentz gauge) :

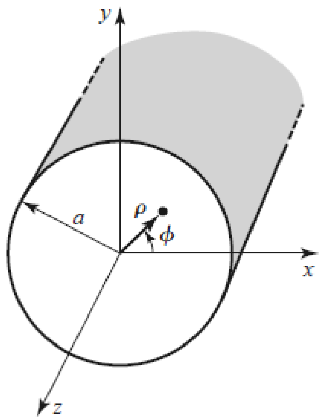
$$\mathbf{A} = A_z \hat{z}$$

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\epsilon_0, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{J}$$

- Helmholtz wave equation In cylindrical coordinates:

$$k_0 = \frac{\omega}{c}$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) A_z + k_0^2 A_z = 0$$



- Separation of variables with arbitrary constant C (complex in general):

$$A_z = C \times J_m(k_\rho \rho)\cos(m\phi)e^{\pm i k_g z}$$

$$-k_g^2 + k_0^2 = k_\rho^2$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(k_\rho^2 - \frac{m^2}{\rho^2}\right)\right] J_m(k_\rho \rho) = 0$$

TM Mode Solution in Circular Waveguide

- Field components can be expressed by A_z alone:

$$\mathbf{B} = \nabla \times \mathbf{A} \longrightarrow B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}, \quad B_\phi = -\frac{\partial A_z}{\partial \rho}, \quad B_z = 0$$

$$\mathbf{E} = \frac{i}{\omega \mu_0 \epsilon_0} \nabla \times \mathbf{B} \longrightarrow E_\rho = -\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\phi}{\partial z}, \quad E_\phi = +\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\rho}{\partial z}$$

$$E_z = \frac{i}{\omega \mu_0 \epsilon_0}[\nabla \times (\nabla \times \mathbf{A})]_z \longrightarrow E_z = \frac{i}{\omega \mu_0 \epsilon_0}\left[\frac{\partial^2 A_z}{\partial z^2} - \nabla^2 A_z\right] = \frac{i}{\omega \mu_0 \epsilon_0} k_\rho^2 A_z$$

- Boundary conditions:

$$E_\phi(\rho = a) = E_z(\rho = a) = B_\rho(\rho = a) = 0$$

$$J_m(k_\rho a) = 0 \longrightarrow k_\rho = \frac{x_{mn}}{a} = \frac{\omega_c}{c}$$

  - x_{mn} : n-th zero of the Bessel function of order m. (e.g., $x_{01} = 2.405$)

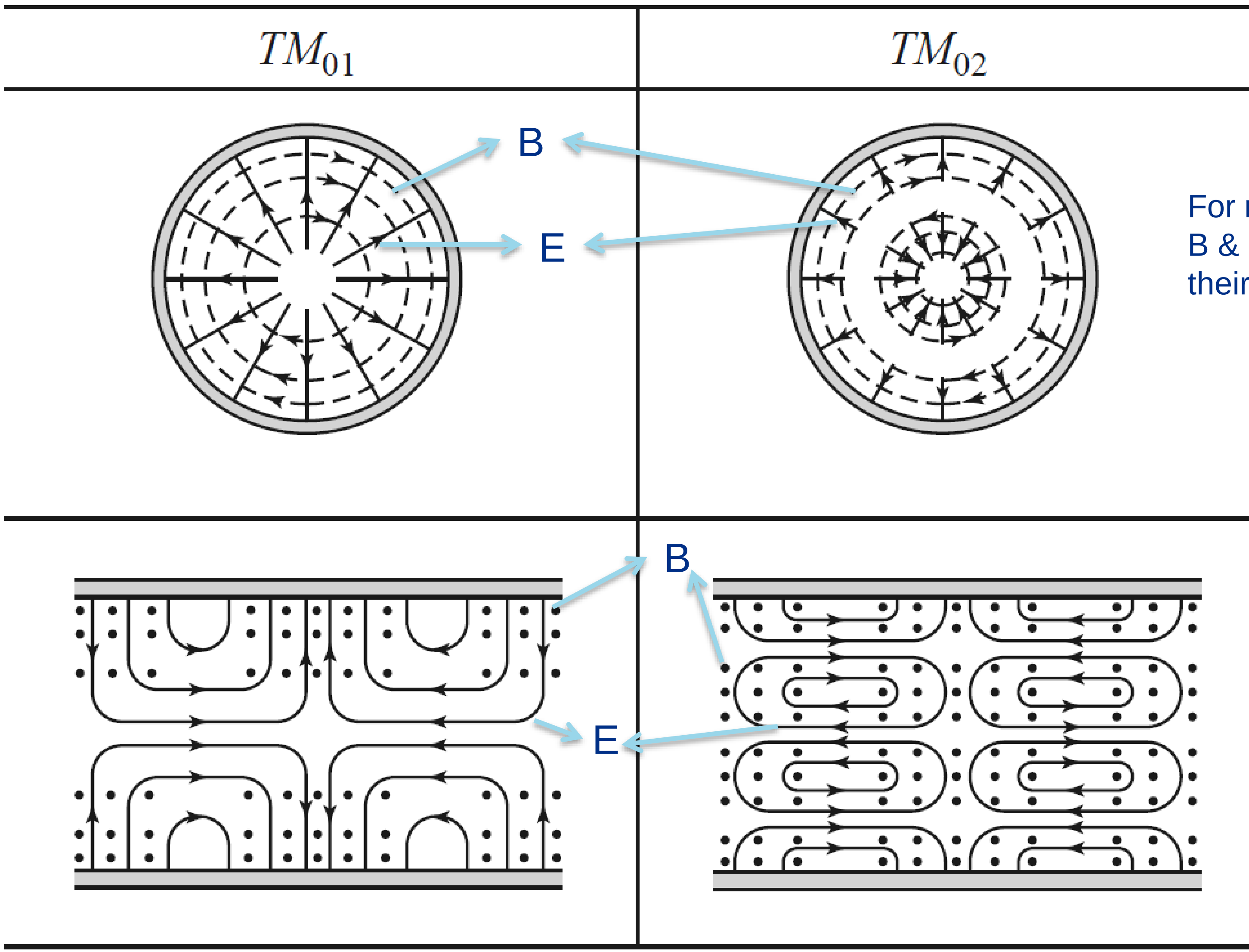
- Dispersion relation for guide propagation constant and wavelength:

cut-off frequency

$$k_g^2 = k_0^2 - k_\rho^2 \longrightarrow \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda_g}\right)^2 + \frac{\omega_c^2}{c^2}$$

[Example]

m=0, n=1 m=0, n=2



For n=2,
B & E change
their signs

Dispersion for a waveguide (Quantitative)

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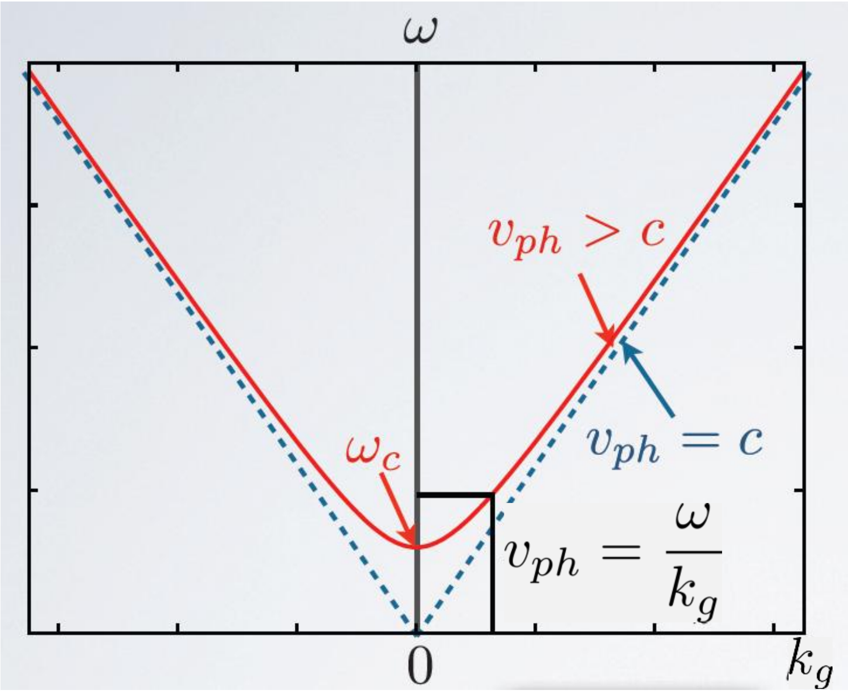
$$\omega^2 = c^2 k_g^2 + \omega_c^2$$

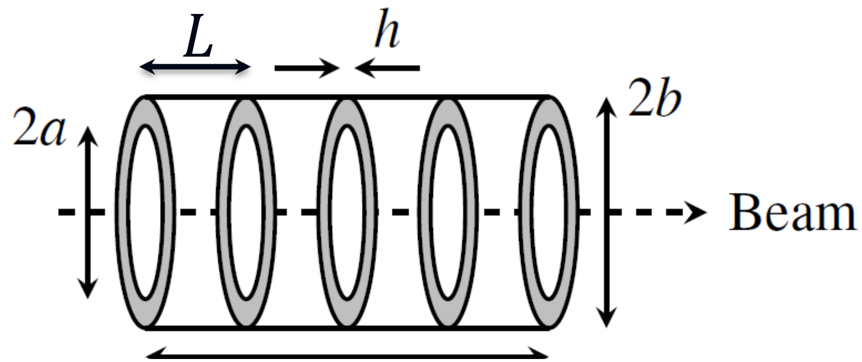


$$v_{ph} v_g = c^2$$

- There is a "cut-off frequency", below which a wave will not propagate. It depends on dimensions.
- At each excitation frequency is associated a phase velocity ω/k_g, the velocity at which a certain phase travels in the waveguide.
- Energy (and information) travel at group velocity $d\omega/dk_g$, which is between 0 and c. This velocity has respect the relativity principle!
- Synchronism with RF (necessary for acceleration) is impossible because a particle would have to travel at $v = v_{ph} > c$!
- To use the waveguide to accelerate particles, we need a "trick" to slow down the wave.

Iris(Disk)-Loaded Waveguide

- In order to slow down the waves in simple waveguide, we introduce some periodic obstacles. Iris acts as a scatter, resulting in a transmitted as well as a reflected wave.



Axial length of the cavity

- The complicated boundary conditions cannot be satisfied by a single mode, but by a whole spectrum of space harmonics.

- From Chap.3.11 of Wangler's textbook [RF Linear Accelerators]: L → l in this book

$$\omega = \frac{2.405c}{b}\sqrt{1+\kappa[1-\cos(k_{gn}L)e^{-\alpha h}]}$$

$$\kappa = \frac{4a^2}{3\pi J_1^2(2.405)b^2 L} \ll 1, \quad \alpha \approx \frac{2.405}{a}, \quad k_{gn} = k_{g0} + \frac{2\pi n}{L}$$

Brillouin Diagram

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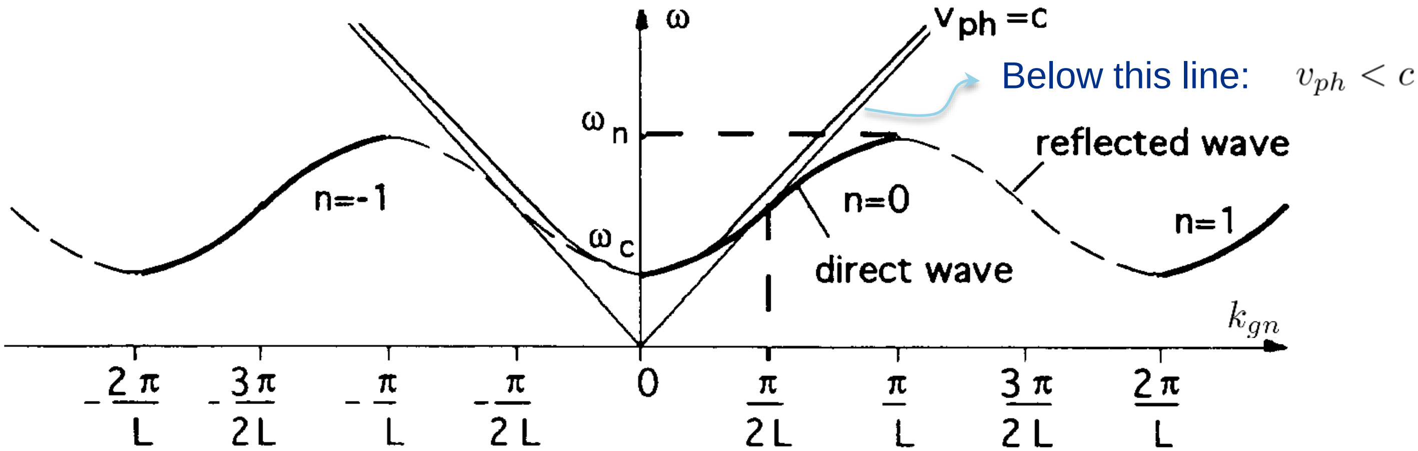


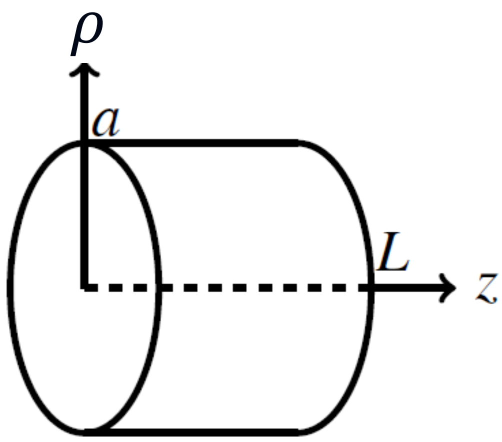
- For a given mode, there is a limited passband of possible frequencies; at both ends of the passband, the group velocity is 0.
- For a given frequency, there is an infinite series of space harmonics ($-\infty < n < +\infty$). All space harmonics have the same group velocity, but different v_{ph}.
- The directed (reflected) wave are characterized by $v_g > 0$ ($v_g < 0$), i.e., the EM energy flows in the $+z$ ($-z$) direction.
- At the end of the waveguide, the EM energy can either be dissipated into a matched load (travelling-wave structure) or be reflected back and forth by shortening end walls (standing-wave structure) → Energy can also be transferred to a particle beam from an standing wave in an RF cavity (next topic).

TM Mode of Pillbox Cavity

미래기반 가속기 전문인력양성 사업단

- We simply superpose two waves in a circular waveguide, one propagating in the positive z direction and the other propagating in the negative z.

$$A_z = C \times J_m(k_\rho \rho)\cos(m\phi)\left(e^{+ik_g z} + e^{-ik_g z}\right) = 2C \times J_m(k_\rho \rho)\cos(m\phi)\cos(k_g z)$$



- Additional boundary conditions at $z = 0$ and $z = L$:

$$E_\rho(z=0) = E_\phi(z=0) = E_\rho(z=L) = E_\phi(z=L) = 0$$

$$E_\rho, E_\phi \propto \sin(k_g z) \;\longrightarrow\; k_g L = p\pi \;\; (p = 0, 1, 2, \cdots)$$

- Dispersion relation: discrete resonance frequency (it was continuous for WG)

$$\frac{\omega^2}{c^2} = \left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2$$

[Example] TM_{010} Mode

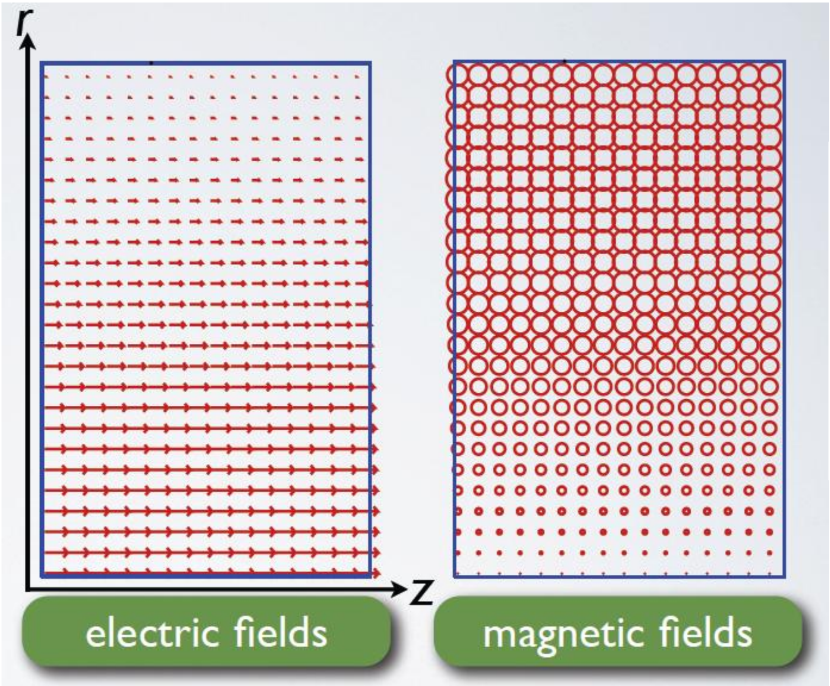
- Simplest and lowest frequency mode: $TM_{mnp} = TM_{010}$

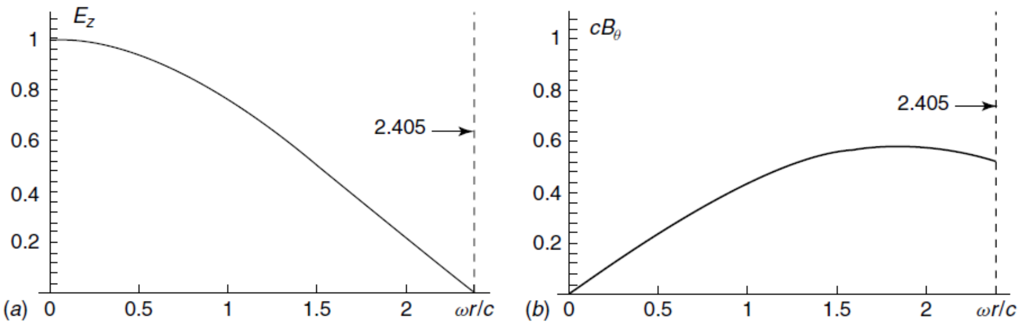
$$k_\rho = \frac{2.405}{a}, \quad \omega = \omega_{010} = \frac{2.405c}{a}$$

- Explicit expression for fields:

$$E_z = E_0 J_0(k_\rho \rho) e^{-i\omega t}, \quad B_\phi = -i\frac{E_0}{c} J_1(k_\rho \rho) e^{-i\omega t}$$

Phase difference





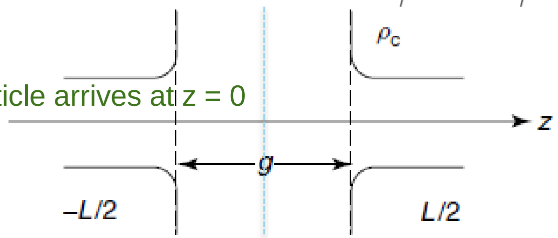
Cavity Parameters: Transit Time Factor

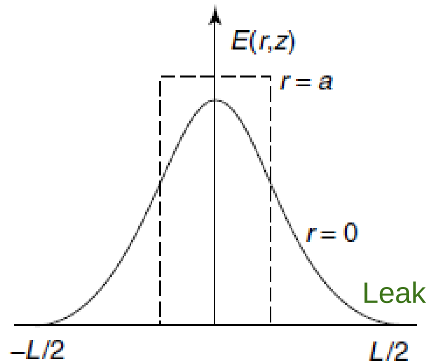
미래기반 가속기 전문인력양성 사업단

- We suppose that the field is symmetric about $z = 0$, and confined within an axial distance L containing the gap, in which velocity change is small.

$$\omega t \approx \omega \frac{z}{v} = 2\pi f \frac{z}{\beta c} = \frac{2\pi z}{\beta \lambda}$$

At t =0, the particle arrives at z = 0





Leak into the beam pipe

$r = 0$

$$\Delta W = q \int_{-L/2}^{L/2} E(0,z)\cos(\omega t + \varphi_s)dz = qV_0 T \cos\varphi_s$$

where

$$V_0 = \int_{-L/2}^{L/2} E(0,z)dz = E_0 L, \quad T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos(\omega t)dz}{V_0} \approx \frac{\int_{-L/2}^{L/2} E(0,z)\cos(2\pi z/\beta\lambda)dz}{V_0}$$

- Accelerating voltage and gradient: Effect of transit time factor (T) is included.

$$V_{acc} = V_0 T, \quad E_0 T \ [\mathrm{MV/m}] = \frac{V_{acc}}{L}$$

Cavity Parameters: Transit Time Factor

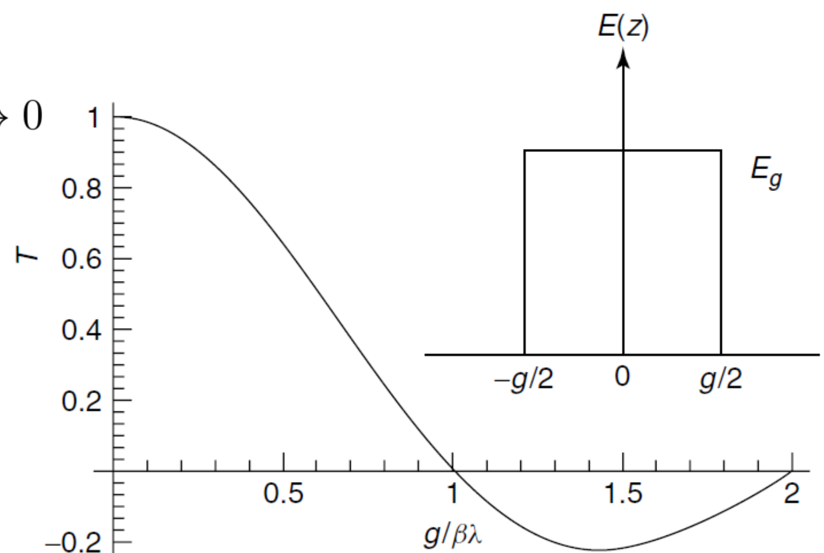
- Physical meaning: ratio of the energy gained in the time-varying RF field to that in a DC field of voltage $V_0\cos(\varphi_s)$.
- Thus, T is a measure of the reduction in the energy gain caused by the sinusoidal time variation of the field in the gap.

Ex] A simple TM_{010} pillbox cavity of length g:

$$E(0,z) = E_g = const., \quad T = \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda}$$

If $g/\beta\lambda \longrightarrow 0$

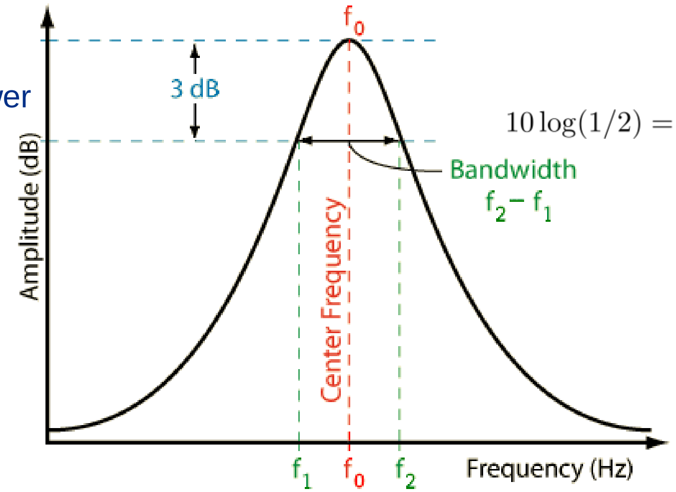
$$\begin{aligned} g &\rightarrow 0 \quad \text{very short gap} \\ \beta\lambda &= v\lambda/c \\ &= vT_{rf} \\ &\rightarrow \infty \quad \text{very fast transit} \end{aligned}$$



If gap is too long, E-field reverses during beam is passing the gap

Cavity Parameters: Quality Factor

- The quality factor Q describes the bandwidth of a resonator and is defined as the ratio of the reactive power (stored energy) to the real power that is dissipated in the cavity walls.



$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f} = \frac{\omega_0 U}{P_d}$$

Ex] For SC cavities, $Q \approx 10^{10} \sim 10^{11}$. Why so high ?

- Filling/Decay time of a cavity: Narrow freq. response → Long time response

$$P_d = -\frac{dU}{dt} = \frac{\omega_0 U}{Q} \quad \longrightarrow U(t) = U_0 e^{-2t/\tau}, \quad \tau = \frac{2Q}{\omega_0}$$

- If the cavity is connected with a power coupler, some power will leak out though the coupler and be dissipated through the external load/waveguide.

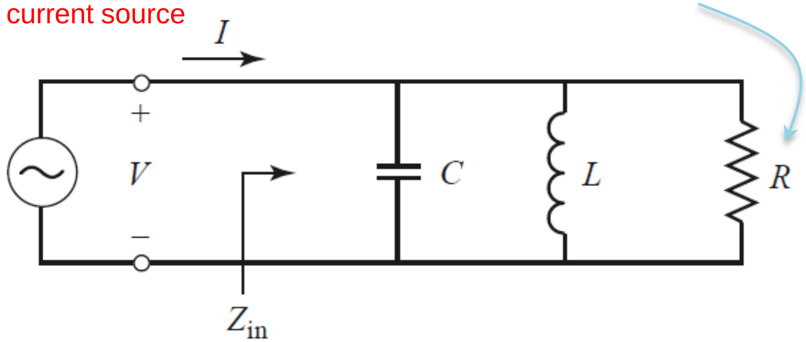
$$Q_{ext} = \frac{\omega_0 U}{P_{ext}}, \quad Q_{loaded} = \frac{\omega_0 U}{P_{ext} + P_{cav}}$$

Resonant Circuit

- A parallel resonant circuit driven by a current generator is the simplest model for describing a single mode of an accelerating cavity (damped driven oscillator).

$$R_s^{circuit} = \frac{V_0^2}{2P_d}$$

Ideal current source



(a)

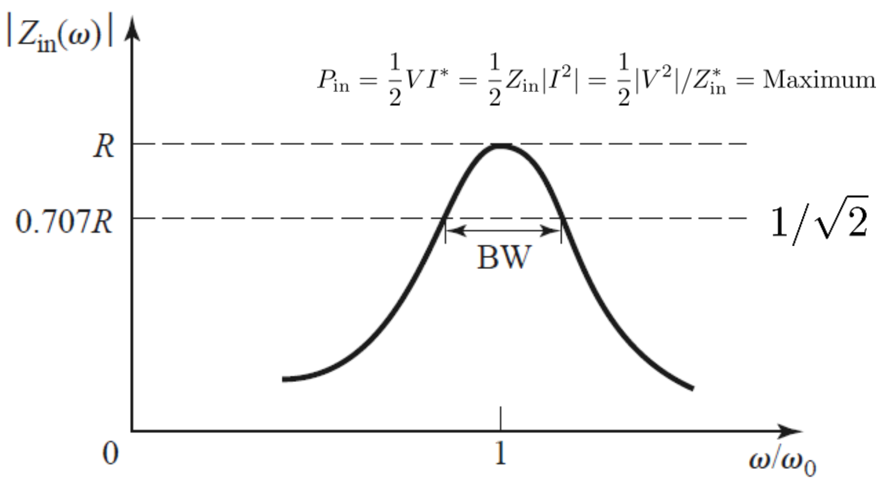
Phase difference

$$I(t) = I_0 e^{j\omega t}, \quad V(t) = V_0 e^{j(\omega t + \phi)}$$

Real amplitude

$$I(t) = C\frac{dV}{dt} + \frac{1}{L}\int V dt + \frac{V}{R}$$

$$V(t) = \underbrace{\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}}_{=Z_{in}} I(t)$$



(b)

– Resonance frequency:

$$\omega_0 = 1/\sqrt{LC}$$

– Total stored energy at resonance ($U_e = U_m$):

$$U = U_e + U_m = CV_0^2/2 = L|I_L|^2/2$$

– Dissipated power:

$$P_d = V_0^2/2R$$

– Quality factor:

$$Q = \omega_0 U/P_d = \omega_0 RC = R/\omega_0 L$$

– Bandwidth:

$$\text{BW} = \frac{1}{Q} = 2\Delta\omega/\omega_0$$

[Pozar “Microwave Engineering” Chap. 6]

Cavity Parameters: Shunt Impedance

- Shunt impedance: A figure of merit that measures the effectiveness of producing an axial voltage V_0 for a given power dissipated P_d.

Don't be confused with surface resistance

$$P_d \propto R_{surf}$$

$$R_s = \frac{V_0^2}{P_d}$$

- Including the transit time factor, we define effective shunt impedance:

$$R_s^{eff} = \frac{(V_0 T)^2}{P_d}$$

- Be careful ! Accelerator community uses different definition of the shunt impedance.

$$R_s^{circuit} = \frac{V_0^2}{2P_d}$$

- R-over-Q: the ratio of R to Q (quality factor), which measures the efficiency of acceleration per unit stored energy U at a given frequency.

$$\left[\frac{R}{Q}\right] = \frac{(V_0 T)^2}{\omega U}$$

  – A single geometric quantity given in Ohms.

Cavity Parameters: Maximum Achievable Gradient

- Empirically derived around 1950, the Kilpatrick limit expresses the relation between the accelerating frequency and maximum achievable accelerating field of any normal conducting cavity:

$$f[\mathrm{MHZ}] = 1.64 E_k[\mathrm{MV/m}]^2 e^{-8.5/E_k[\mathrm{MV/m}]}$$

- After improving surface quality and cleanness to avoid RF breakdown, a considerable increase of achievable accelerating gradients has been made. In particular, Wang and Loew's empirical formula, devised in 1997, suggests the following behaviors:

$$E[\mathrm{MV/m}] = 220 f[\mathrm{GHz}]^{1/3}$$

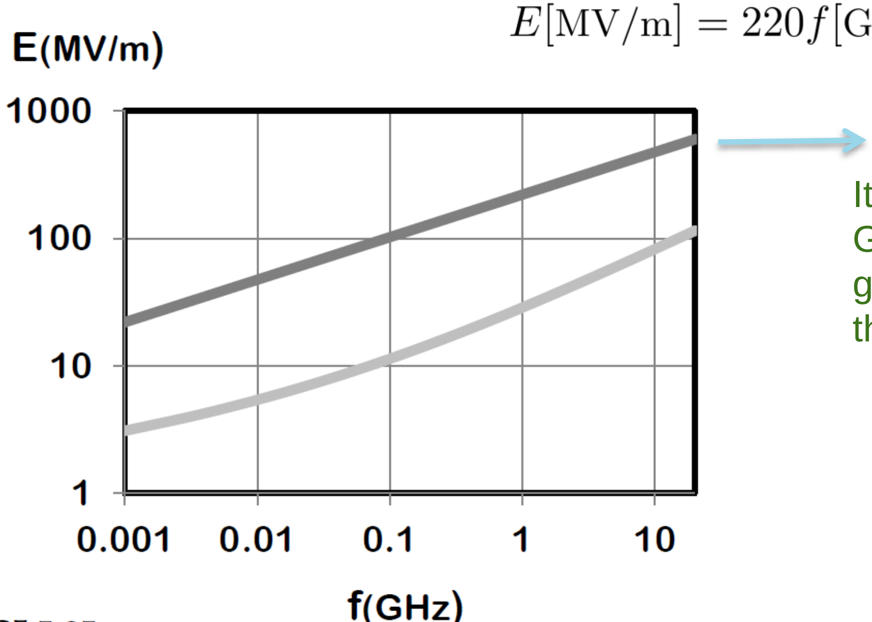
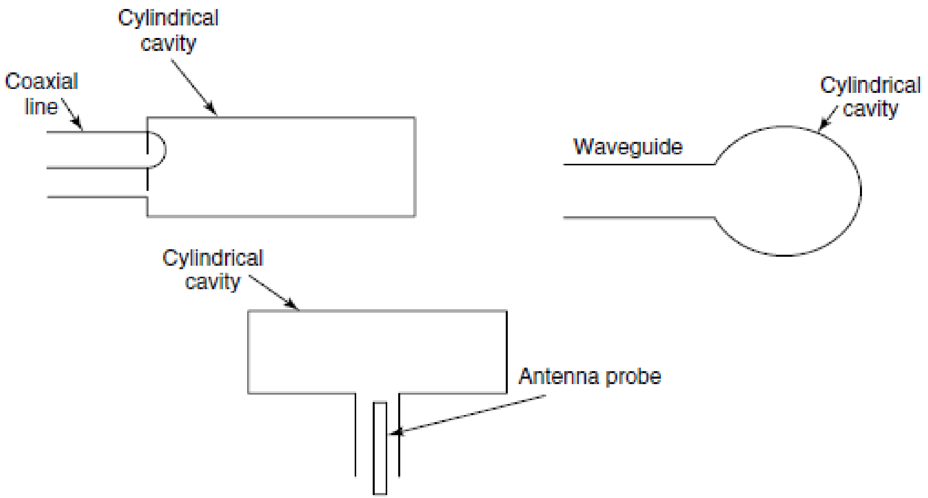


FIGURE 5.25
Breakdown Kilpatrick limit (lower curve) and Wang–Loew limit (upper curve).

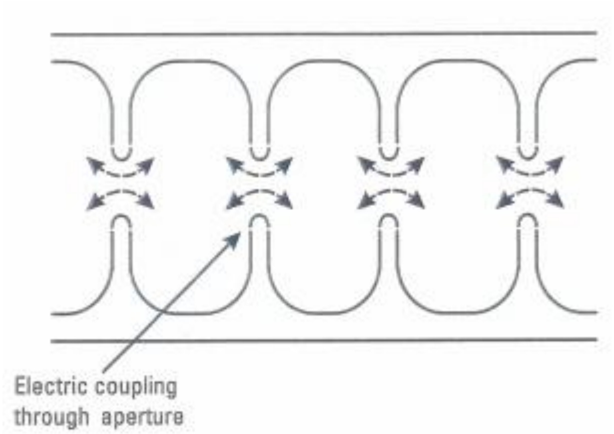
It was observed that at higher (multi-tens of GHz) frequency regimes, the maximum gradients appear to be rather independent of the frequency.

Coupling to cavities

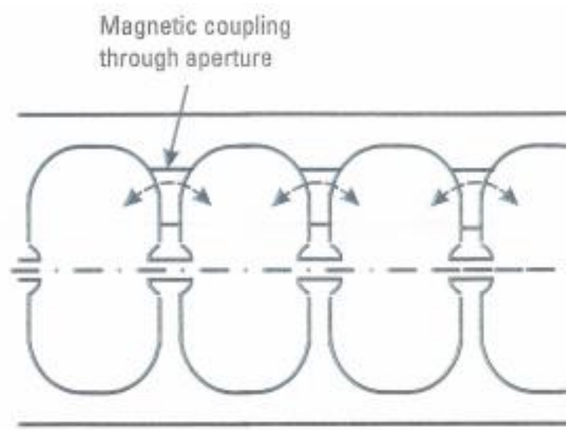
Methods of coupling to cavities



Coupling between cavities



The cavities are coupled by the electric field shared by every two adjacent cavities through the coupling apertures along the common axis.

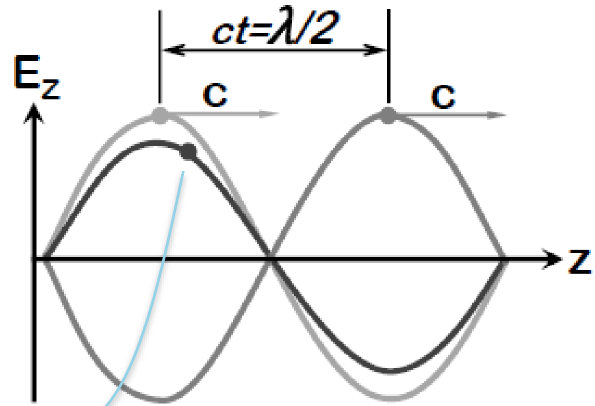


We have to open slots or apertures in the common wall between every two adjacent cavities in the regions of high magnetic fields.

Acceleration in RF Structure

- Standing wave: The particle bunch in a standing wave observes the electric field with a varying function of time as

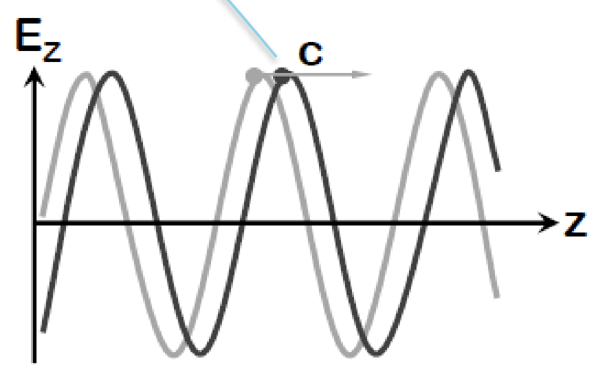
$$E_z = E_0 \cos(\omega t + \varphi_s) \sin(kz)$$



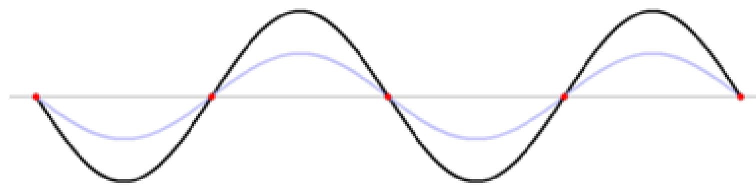
Particles at later time
(black dots)

- Travelling wave: The particles in an appropriately synchronized travelling wave experience a constant electric field

$$E_z = E_0 \cos(\omega t - kz) = E_0 \cos(\varphi_s)$$



Some Animation

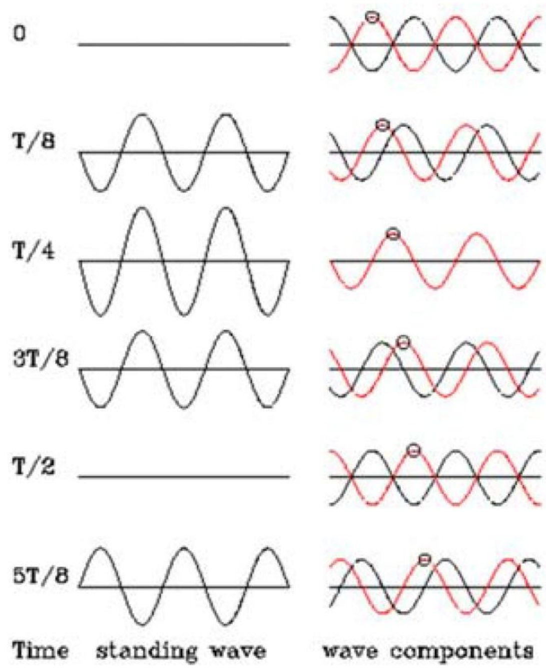


two oppositely-travelling waves

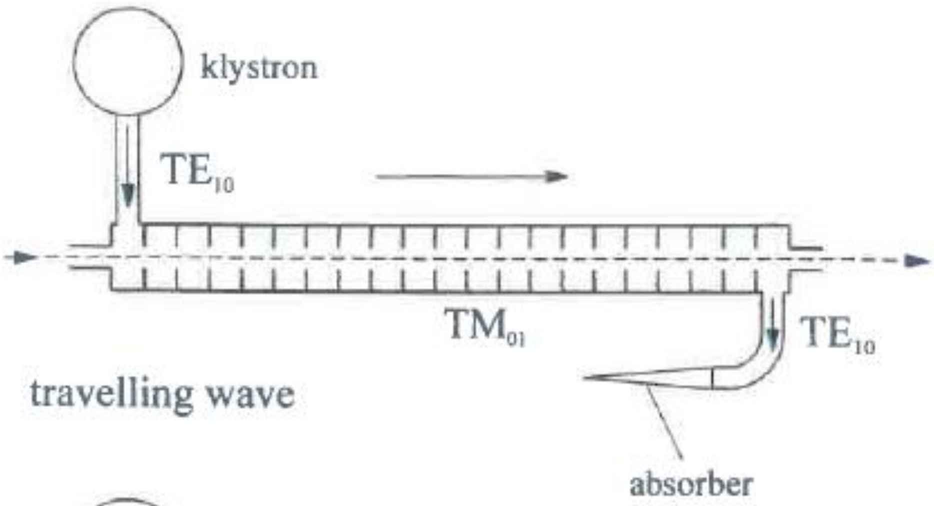
one standing wave

Comments on the SW and TW Structure

- The standing wave modes are generated by the sum of 2 traveling waves in opposite directions.
- Since only the forward wave can accelerate the beam, the shunt impedance (effectiveness of producing axial voltage for a given power dissipated) is ½ of that of the travelling wave structure.
- The standing wave could accelerate oppositely charged beams traveling in opposite directions.

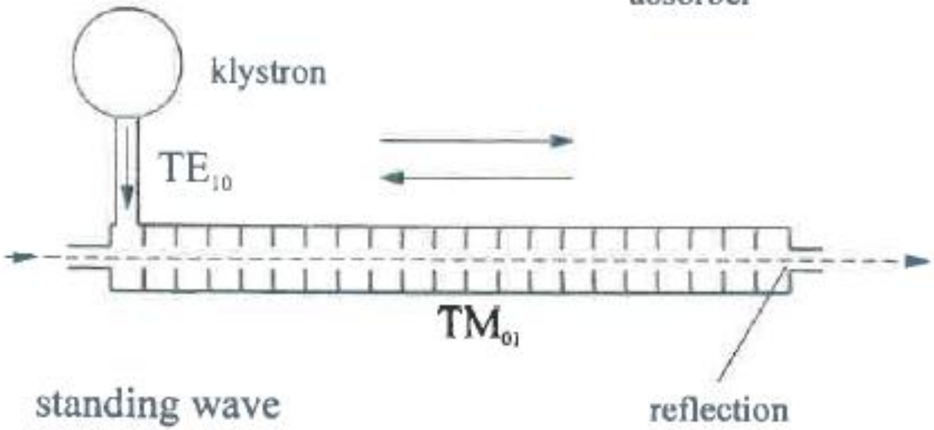


Comments on the SW and TW Structure



Short pulses, High frequency (≥3GHz)
Gradients: 10~20 MeV/m
Used for electrons at v~c

- Comparable RF efficiency
- No definite reasons to prefer one or the other



Long pulses
Gradients: 2~5 MeV/m
Used for ions and electrons at all energies
→ SW is more suitable in certain cases, such as for superconducting cavities.