Derivation of Bunch lengthening factor in double RF system

Jun-hyoung Kim
DANE, POSTECH

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Outline

Recall) Introduction to 3HC

1. Phase space density

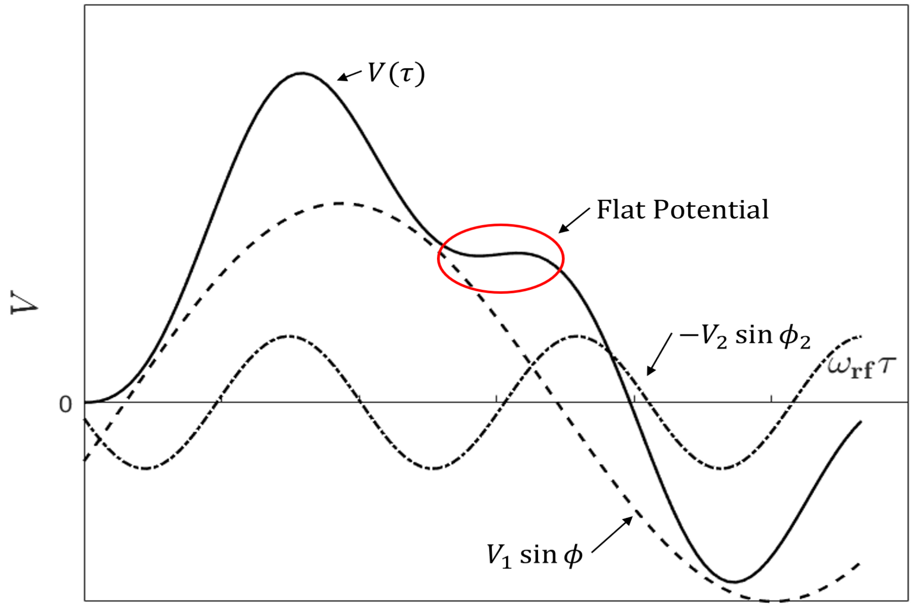
2. Bunch length (without 3HC)

3. Bunch length in the presence of 3HC

Recall) Introduction to HHC

- Harmonic cavity can *improve the beam quality through bunch size lengthening* which includes providing *Landau damping*, suppressing *coupled bunch instability* and *microwave instability*, enhancing the *beam current per bunch* besides the *lifetime improvement*. [Hou]

$$V(\tau) = V_1[\sin(\omega_{\mathrm{rf}}\tau + \phi_{1\mathrm{s}}) - r\sin(m\omega_{\mathrm{rf}}\tau + \phi_{2\mathrm{s}})] - \frac{U_0}{e}$$



1. Phase space density $\rho(\vec{q}, \vec{p}, t)$

- Liouville theory

$$\frac{\partial \rho}{\partial t} + [\rho, \mathcal{H}] = 0$$

- Time evolution of phase space density

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = \frac{\partial \rho}{\partial t} + [\rho, \mathcal{H}] = 0$$

$\therefore$ ρ is a constant of motion

- Normalized phase space density = pdf $\psi(\vec{q}, \vec{p}, t)$

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + [\psi, \mathcal{H}] = 0$$

- In equilibrium, $\frac{\partial \psi}{\partial t} = 0 \quad \rightarrow \psi = \psi(\mathcal{H})$

2. Bunch length (w/o HHC)

- Assume $\psi \propto \exp\left[-\frac{1}{2}\left(\frac{\Delta E}{\sigma_E}\right)^2\right] \propto \exp\left[-\frac{1}{2}\left(\frac{\delta}{\sigma_\delta}\right)^2\right]$
- Also, $\psi = \psi(\mathcal{H})$, while

$$\mathcal{H} = \frac{1}{2}h\omega_0\eta\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}\{\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\}$$

- Therefore,

$$\psi(\phi,\delta) = C\exp\left[-A\left(\frac{1}{2}h\omega_0\eta\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}\{\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\}\right)\right]$$

$$= C\exp\left[-\frac{\delta^2 + \frac{eV}{\pi h\eta\beta^2 E}\{\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\}}{2\sigma_\delta^2}\right]$$

2. Bunch length (w/o HHC)

- Short bunch approximation $\varphi = \phi - \phi_s \ll 1$

$$\psi(\phi, \delta) = C \exp\left[-\frac{\delta^2 + \left(\frac{Q_s}{h\eta}\right)^2 \varphi^2}{2\sigma_\delta^2}\right] \equiv \frac{1}{2\pi\sigma_\delta\sigma_\phi} \exp\left[-\frac{1}{2}\left(\frac{\varphi^2}{\sigma_\phi^2} + \frac{\delta^2}{\sigma_\delta^2}\right)\right]$$

$$\therefore \quad \sigma_\phi = \frac{h\eta}{Q_s}\sigma_\delta$$

3. Bunch length in the presence of 3HC

- Hamiltonian

$$\mathcal{H}(\tau,\delta) = \frac{\eta}{2}\delta^2 + \frac{eV}{2\pi h\beta^2 E}\left[\cos(\omega_{\mathrm{rf}}\tau + \phi_{1\mathrm{s}}) - \cos\phi_{1\mathrm{s}} - \frac{r}{m}\cos(m\omega_{\mathrm{rf}}\tau + \phi_{2\mathrm{s}}) + \frac{r}{m}\cos\phi_{2\mathrm{s}} + \omega_{\mathrm{rf}}\tau\sin\phi_{0\mathrm{s}}\right]$$

- Short-bunch approximation

$$\mathcal{H}(\tau,\delta) = \frac{\eta}{2}\delta^2 + \frac{eV}{2\pi h\beta^2 E}\left[\frac{1}{4!}(\omega_{\mathrm{rf}}\tau)^4(rm^3\cos\phi_{2\mathrm{s}} - \cos\phi_{1\mathrm{s}}) + O(\tau^5)\right]$$

$$= \frac{\eta}{2}\left[\delta^2 + \left(\frac{Q_{1\mathrm{s}}}{h\eta}\right)^2\frac{m^2-1}{12}(\omega_{\mathrm{rf}}\tau)^4\right]$$

$Q_{1\mathrm{s}}$: synchrotron tune for main RF cavity in the presence of 3HC

3. Bunch length in the presence of 3HC

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(\varphi,\delta)d\varphi\, d\delta = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_\delta}\exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)d\delta\int_{-\infty}^{\infty}C_2\exp\left[-\left(\frac{Q_{1s}}{h\eta\sigma_\delta}\right)^2\frac{m^2-1}{24}\varphi^4\right]d\varphi$$

$$= C_2\int_{-\infty}^{\infty}\exp\left[-\left(\frac{Q_{1s}}{h\eta\sigma_\delta}\right)^2\frac{m^2-1}{24}\varphi^4\right]d\varphi$$

Take $\lambda^2 = \left(\frac{Q_{1s}}{h\eta\sigma_\delta}\right)^2\frac{m^2-1}{24}$,

$$C_2 = \frac{2\lambda^{\frac{1}{2}}}{\Gamma\left(\frac{1}{4}\right)}, \qquad \psi(\varphi) = C_2 e^{-\lambda^2\varphi^4}$$

3. Bunch length in the presence of 3HC

- RMS Bunch length

$$\sigma_\phi = \sqrt{\int_{-\infty}^{\infty} \varphi^2 \psi(\varphi) d\varphi} = \left[\left(\frac{h\eta}{Q_{1s}}\right)^2 \frac{24}{m^2-1}\right]^{\frac{1}{4}} \sigma_\delta^{\frac{1}{2}}$$

- Bunch lengthening factor

$$BLF \equiv \frac{\sigma_\phi(\text{with HHC})}{\sigma_\phi(\text{without HHC})} = \frac{\left[\left(\frac{h\eta}{Q_{1s}}\right)^2 \frac{24}{m^2-1}\right]^{\frac{1}{4}} \sigma_\delta^{\frac{1}{2}}}{\frac{h\eta}{Q_s}\sigma_\delta} = \sqrt{\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}} \left[\left(\frac{Q_s}{h\eta}\right)^2 \frac{24}{m^2-1} \frac{\cos\phi_{0s}}{\cos\phi_{1s}}\right]^{\frac{1}{4}} \frac{1}{\sqrt{\sigma_\delta}}$$

- For 3HC in PLS-II, $BLF = 3.08$